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Rotational Motion Control by Feedback with Minimum L_1-Norm

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Introduction

CONTROL of the rotational motion of a spacecraft is an important issue in spacecraft docking, in space missions such as inspection and repair of spacecraft in orbit, in capture and removal of space debris [1], and in formation flight with attitude alignment [2–4].

Consider a spacecraft and an arbitrary torque-free motion, which is regarded as a reference motion. Our control problem is to find a feedback control that steers the spacecraft asymptotically to the reference motion. If the reference motion is stationary, this becomes a standard attitude control problem [5,6]. If the reference motion is a spin, it covers the reorientation of spinning spacecraft [7,8] as well as the tracking problem [9,10].

Here, feedback controllers are designed from the minimum fuel point of view, and L_1-norm is taken as a main performance index. This is because torque is proportional to the magnitude of thrust, and fuel consumption is proportional to the L_1-norm of the input torque. To solve this problem, the dynamics of attitude motion are described by the kinematics differential equations of quaternions and the Euler's rotational equations of motion [11]. To design feedback controllers, the linearized equations of the relative motion are employed. In the relative orbit transfer problem along a circular orbit and an eccentric orbit, linearized equations are known as Hill–Clohessy–Wiltshire (HCW) equations and Tschauner–Hempel (TH) equations, respectively, and they have been extensively used for proximity operations such as rendezvous, docking, inspection, and repair [12,13]. For rotational motion control, the linearized equations of the relative motion are equally important, and their properties are potentially useful for many future operations. If the reference motion is a spin about a principal axis, the linear system is time-invariant. For axisymmetric spacecraft, it becomes periodic if the reference motion is a general spin (i.e., a spin about an arbitrary axis).

In this paper, two useful properties of the linearized system are proved. The first one is a Floquet representation of the state transition matrix of the linear system [14]. It is explicitly given in terms of a constant system matrix and an orthogonal transformation matrix. The transformed system is time-invariant, and any linear control theory can be applicable.

The second property is null controllability with vanishing energy (NCVE). It is a property for a linear system that any state of the system can be steered to the origin with an arbitrarily small amount of control energy in the L_2 (square integral) sense at the expense of the control duration. Two sets of necessary and sufficient conditions for NCVE are known [15,16]. A time-invariant system (A, B) is null controllable with vanishing energy (also NCVE) if and only if it is controllable and all eigenvalues of A have nonpositive real parts. The second condition can be replaced by that for which the maximal solution of the singular algebraic Riccati equation is zero. A periodic system $[A(t), B(t)]$ is NCVE if and only if it is controllable on some interval and all eigenvalues of the monodromy matrix have a modulus less than 1 [17]. Replacement of the algebraic Riccati equation by the differential Riccati equation gives the second set of necessary and sufficient conditions. Note that the definition of NCVE implies that the L_2-norm of a controller does not represent fuel consumption or any physical energy. Note also that systems with a NCVE property do not have unstable modes, which justifies the use of the linearized system of relative rotational motion.

Although L_1-norm of the control is a right performance index, there is no good control theory that is directly related to it. NCVE assures that the L_2-norm of the feedback control based on the algebraic (or differential) Riccati equation of the linear-quadratic-regulator (LQR) theory converges to zero as the penalty matrix on the state decreases to zero. Shibata and Ichikawa [16] showed that HCW equations and TH equations with three independent thrusters are NCVE. By numerical simulations, they also showed that the L_1-norm of the feedback controller decreases monotonically to a positive constant as the penalty matrix on the state decreases to zero. Thus, the LQR theory can be used to design L_1 suboptimal feedback controllers, and NCVE makes their L_1-norms relatively small.

The NCVE property of the linearized system of relative rotational motion readily follows from the Floquet representation theorem. It is then shown that the differential Riccati equation can be reduced to an algebraic Riccati equation, which simplifies the design of feedback controllers. By numerical simulations, the behavior of the L_1-norm of the feedback controller is examined. For rotational motion, there is a new feature, in that the L_1-norm has a global minimum. Hence, L_1 optimal and suboptimal feedback controllers can be designed. In simulations, the initial error of the angular velocity is assumed to be 10%, and the initial error of the alignment of the axis of symmetry is 90 deg.

Two cases of the reference motion are considered: a general spin and a pure spin about the axis of symmetry. In each case, feedback controllers based on the algebraic Riccati equation are designed and applied to the original nonlinear system of relative rotational motion. As for performance indices, the L_1-norm of the controller, the maximum value of the control input, and the settling time are calculated. Although the error of the axis alignment is large, the controller works well, and the rotational motion of the spacecraft converges to the reference motion smoothly.

Attitude control of spacecraft has been extensively studied in the literature. However, to the best of our knowledge, neither the Floquet representation of the linearized system nor the minimum L_1-norm problem have been discussed. The notion of NCVE was first introduced by Priola and Zabczyk [15] in 2003 for infinite-dimensional systems. It is extended to discrete-time systems and

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periodic systems by Ichikawa [17]. The NCVE property of HCW equations is also used for the relative orbit transfer problem by impulse control [18].

Relative Rotational Equations of Motion

Consider a spacecraft with body-fixed reference frame R_B with basis vectors ($\boldsymbol{b}_1$, $\boldsymbol{b}_2$, and $\boldsymbol{b}_3$). In this paper, the rotational motion is described by the kinematic differential equations and the equations of motion, given by

$$\dot{\boldsymbol{q}} = \Omega(\boldsymbol{\omega})\boldsymbol{q} \tag{1}$$

$$J\dot{\boldsymbol{\omega}} = -S(\boldsymbol{\omega})J\boldsymbol{\omega} + \boldsymbol{\tau} \tag{2}$$

where $\boldsymbol{q} \in \boldsymbol{R}^4$ is the quaternion of the spacecraft with respect to a given inertial reference frame R_I, $\boldsymbol{\omega} \in \boldsymbol{R}^3$ is the angular velocity vector of the spacecraft with respect to R_B, J is the inertia matrix, $\boldsymbol{\tau} \in \boldsymbol{R}^3$ is an external torque, $S(\boldsymbol{\omega}) \in \boldsymbol{R}^{3\times 3}$ is the cross-product matrix of $\boldsymbol{\omega}$, and $\Omega(\boldsymbol{\omega}) \in \boldsymbol{R}^{4\times 4}$ is the skew-symmetric matrix of the kinematic differential equation [11]. Throughout the paper, it is assumed that R_B consists of the principal axes so that $J = \mathrm{diag}\{J_1, J_2, J_3\}$.

Now introduce a reference motion of the spacecraft given by $\bar{\boldsymbol{q}}$ and $\bar{\boldsymbol{\omega}}$, which is a free motion of Eqs. (1) and (2). The reference frame of this motion is denoted by $\bar{R}_B$. In this paper, feedback controls that steer the spacecraft asymptotically to the reference motion will be sought, and the main performance index is L_1-norm, which represents fuel consumption. For this purpose, the equations of relative motion

$$\dot{\boldsymbol{q}}_r = \Omega(\boldsymbol{\omega}_r)\boldsymbol{q} \tag{3}$$

$$\dot{\boldsymbol{\omega}}_r = S(\boldsymbol{\omega}_r)C(\boldsymbol{q}_r)\bar{\boldsymbol{\omega}} - C(\boldsymbol{q}_r)\dot{\bar{\boldsymbol{\omega}}} - J^{-1}[S(\boldsymbol{\omega}_r) + S(C(\boldsymbol{q}_r)\bar{\boldsymbol{\omega}})]J[\boldsymbol{\omega}_r + C(\boldsymbol{q}_r)\bar{\boldsymbol{\omega}}] + J^{-1}\boldsymbol{\tau} \tag{4}$$

are used, where $\boldsymbol{q}_r$ is the quaternion of the reference frame R_B relative to $\bar{R}_B$ given by $\boldsymbol{q}_r = Q'(\bar{\boldsymbol{q}})\boldsymbol{q}$, $Q(\boldsymbol{q}) \in \boldsymbol{R}^{4\times 4}$ is the quaternion matrix [11], $\boldsymbol{\omega}_r$ is the relative angular velocity vector $\boldsymbol{\omega}_r = \boldsymbol{\omega} - C(\boldsymbol{q}_r)\bar{\boldsymbol{\omega}}$, and $C(\boldsymbol{q}_r) \in \boldsymbol{R}^{3\times 3}$ is the direction cosine matrix of the reference frame R_B relative to $\bar{R}_B$.

To design a control law that brings the state of the relative motion to zero, Eqs. (3) and (4) are linearized at the origin $\boldsymbol{q}_{r0} = [0, 0, 0, 1]$ and $\boldsymbol{\omega}_{r0} = [0, 0, 0]$. Then the system of linearized equations is given by

$$\dot{x} = Ax + Bu \tag{5}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \\ 2k_1(\bar{\omega}_3^2 - \bar{\omega}_2^2) & 2(k_1 + k_3)\bar{\omega}_1\bar{\omega}_2 & -2(k_1 + k_2)\bar{\omega}_3\bar{\omega}_1 & 0 & (k_1 + 1)\bar{\omega}_3 & (k_1 - 1)\bar{\omega}_2 \\ -2(k_2 + k_3)\bar{\omega}_1\bar{\omega}_2 & 2k_2(\bar{\omega}_1^2 - \bar{\omega}_3^2) & 2(k_2 + k_1)\bar{\omega}_2\bar{\omega}_3 & (k_2 - 1)\bar{\omega}_3 & 0 & (k_2 + 1)\bar{\omega}_1 \\ 2(k_3 + k_2)\bar{\omega}_3\bar{\omega}_1 & -2(k_3 + k_1)\bar{\omega}_2\bar{\omega}_3 & 2k_3(\bar{\omega}_2^2 - \bar{\omega}_1^2) & (k_3 + 1)\bar{\omega}_2 & (k_3 - 1)\bar{\omega}_1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} O_{3\times 3} \\ J^{-1} \end{bmatrix} \tag{6}$$

where

$$\boldsymbol{x} = [q_{r1}, q_{r2}, q_{r3}, \omega_{r1}, \omega_{r2}, \omega_{r3}]'$$

$\boldsymbol{u} = \boldsymbol{\tau}$, and k_i ($i = 1, 2, 3$) are the ratios of the principal moments of inertia, defined by

$$k_1 = \frac{J_2 - J_3}{J_1} \qquad k_2 = \frac{J_3 - J_1}{J_2} \qquad k_3 = \frac{J_1 - J_2}{J_3}$$

Because $\dot{q}_{r4} = 0$ and q_{r4} is independent of the rest, it has been eliminated in the linearized equations. Note that A is independent of $\bar{\boldsymbol{q}}$. If $\dot{\bar{\boldsymbol{\omega}}} \neq 0$, then A becomes a function of time. However, if the reference motion is a spin about a principal axis, then A is a constant matrix. Note also that (A, B) is always controllable because of the special structure of the (1, 2) block of A and the second block of B.

Floquet Representation and NCVE for Axisymmetric Spacecraft

In this paper, the spacecraft is assumed to be axisymmetric. For definiteness, the inequality $J_1 = J_2 = J > J_3$ for principal moments of inertia is assumed. Thus, the minor axis $\boldsymbol{b}_3$ is the axis of symmetry, and the spacecraft is rod-shaped (prolate). Then $k_1 = -k_2 = k$ with $0 < k < 1$, and $k_3 = 0$. The reference motion is governed by

$$\dot{\bar{\omega}}_1 = \mu\bar{\omega}_2, \qquad \dot{\bar{\omega}}_2 = -\mu\bar{\omega}_1, \qquad \dot{\bar{\omega}}_3 = 0 \tag{7}$$

where $\mu = k\Omega_3$ is the transverse frequency, and $\bar{\omega}_3 = \Omega_3$ is the spin rate.

Substitute $k_1 = -k_2 = k$ and $k_3 = 0$ into the system matrix (6), then it becomes

$$A = \begin{bmatrix} 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \\ 2k(\bar{\omega}_3^2 - \bar{\omega}_2^2) & 2k\bar{\omega}_1\bar{\omega}_2 & 0 & 0 & (k + 1)\bar{\omega}_3 & (k - 1)\bar{\omega}_2 \\ 2k\bar{\omega}_1\bar{\omega}_2 & 2k(\bar{\omega}_3^2 - \bar{\omega}_1^2) & 0 & -(k + 1)\bar{\omega}_3 & 0 & -(k - 1)\bar{\omega}_1 \\ -2k\bar{\omega}_3\bar{\omega}_1 & -2k\bar{\omega}_2\bar{\omega}_3 & 0 & \bar{\omega}_2 & -\bar{\omega}_1 & 0 \end{bmatrix} \tag{8}$$

The solution of Eq. (7) is given by

$$\bar{\boldsymbol{\omega}}(t) = [\Omega_{12}\sin(\mu t + \phi), \Omega_{12}\cos(\mu t + \phi), \Omega_3]'$$

where $\Omega_{12} = [\omega_1^2(0) + \omega_2^2(0)]^{1/2}$ is the size of the transverse angular rate, and ϕ is the phase given by $\tan\phi = \omega_1(0)/\omega_2(0)$. Thus, A is a periodic function with period $p = 2\pi/\mu$ and is given by

$$A(t)=\begin{bmatrix} 0 & 0 & 0 & 1/2 & 0 & 0\\ 0 & 0 & 0 & 0 & 1/2 & 0\\ 0 & 0 & 0 & 0 & 0 & 1/2\\ 2k(\Omega_3^2-\Omega_{12}^2c^2) & 2k\Omega_{12}^2sc & 0 & 0 & (k+1)\Omega_3 & (k-1)\Omega_{12}c\\ 2k\Omega_{12}^2sc & 2k(\Omega_3^2-\Omega_{12}^2s^2) & 0 & -(k+1)\Omega_3 & 0 & -(k-1)\Omega_{12}s\\ -2k\Omega_{12}\Omega_3 s & -2k\Omega_{12}\Omega_3 c & 0 & \Omega_{12}c & -\Omega_{12}s & 0 \end{bmatrix} \quad (9)$$

where $s=\sin(\mu t+\phi)$ and $c=\cos(\mu t+\phi)$. Let $\Phi(t,s)$ be the state transition matrix of the system

$$\dot{\boldsymbol{x}}(t)=A(t)\boldsymbol{x}(t)+B\boldsymbol{u}(t), \qquad \boldsymbol{x}(t_0)=\boldsymbol{x}_0 \quad (10)$$

Then the following Floquet representation [14] of $\Phi(t,s)$ holds. A remarkable consequence is that by a simple rotation matrix, system (10) is reduced to a time-invariant system.

Theorem 1:

$$\Phi(t,s)=P(\mu t)\exp\bar{A}(t-s)P'(\mu s) \quad (11)$$

where $P(t)$ is an orthogonal matrix given by

$$P(t)=\begin{bmatrix} \cos t & \sin t & 0 & & & \\ -\sin t & \cos t & 0 & & O_{3\times3} & \\ 0 & 0 & 1 & & & \\ & & & \cos t & \sin t & 0\\ & O_{3\times3} & & -\sin t & \cos t & 0\\ & & & 0 & 0 & 1 \end{bmatrix} \quad (12)$$

and $\bar{A}=P(\phi)\bar{A}_0P'(\phi)$, where $\bar{A}_0$ is a constant matrix given by

$$\bar{A}_0=\begin{bmatrix} 0 & -k\Omega_3 & 0 & 1/2 & 0 & 0\\ k\Omega_3 & 0 & 0 & 0 & 1/2 & 0\\ 0 & 0 & 0 & 0 & 0 & 1/2\\ 2k(\Omega_3^2-\Omega_{12}^2) & 0 & 0 & 0 & \Omega_3 & (k-1)\Omega_{12}\\ 0 & 2k\Omega_3^2 & 0 & -\Omega_3 & 0 & 0\\ 0 & 2k\Omega_{12}\Omega_3 & 0 & \Omega_{12} & 0 & 0 \end{bmatrix}$$

and the set of eigenvalues of $\bar{A}$ is $\sigma(\bar{A})=\{0(\text{quadruple}),\pm i\nu\}$, where $\nu=[\Omega_{12}^2+(k-1)^2\Omega_3^2]^{1/2}\}$.

Proof. Note that the diagonal block $P_0(t)\in\boldsymbol{R}^{3\times3}$ of $P(t)$ is a rotation matrix, and $P(\mu t)$ is p-periodic with $P(0)=I$. By direct calculation, one can show the equality:

$$\bar{A}_0=P(\mu t+\phi)'[A_3(t)P(\mu t+\phi)-\dot{P}(\mu t+\phi)] \quad (13)$$

In view of the property $P(\phi+\psi)=P(\phi)P(\psi)$, the preceding equality is reduced to

$$\bar{A}=P(\mu t)'[A_3(t)P(\mu t)-\dot{P}(\mu t)] \quad (14)$$

Now by the transformation $x=P(\mu t)z$, Eq. (10) is written as

$$\dot{z}(t)=\bar{A}z(t)+P'(\mu t)B\boldsymbol{u}(t) \quad (15)$$

which proves Eq. (11). The eigenvalues of $\bar{A}$ are obtained from the characteristic equation of $\bar{A}_0$: that is,

$$|\lambda I-\bar{A}_0|=\lambda^4[\lambda^2+\Omega_{12}^2+(k-1)^2\Omega_3^2]$$

Because $P'(\mu t)B=BP_0'(t)$, system (15) is equivalent to

$$\dot{z}(t)=\bar{A}z(t)+B\boldsymbol{v}(t) \quad (16)$$

where $\boldsymbol{v}(t)=P_0'(\mu t)\boldsymbol{u}(t)$.

Note that $\bar{A}$ and $P(\mu t)$ have been derived as follows. First, $\Phi(t,0)$ is directly calculated. Then $\bar{A}p$ is given as a logarithm of the monodromy matrix $M=\Phi(p,0)$, so that $\exp\bar{A}p=M$. Finally, $P(\mu t)$ is defined by

$$P(\mu t)=\Phi(t,0)\exp(-\bar{A}t) \quad (17)$$

Because $\sigma(M)=\sigma(\exp\bar{A}p)=\exp\{\sigma(\bar{A})p\}$ holds,

$$\sigma(M)=\{1(\text{quadruple}),\pm\exp(i2\pi\nu/\mu)\} \quad (18)$$

In view of Eq. (11), the elements of the state transition matrix $\Phi(t,s)$ contain products of $\{\cos\mu t,\sin\mu t\}$ and $\{\cos\nu t,\sin\nu t\}$.

Now the NCVE property of system (10) will be examined. This property makes the L_1-norm of the LQR controller relatively small. Recall that a linear system is null-controllable with vanishing energy [respectively, controllable with vanishing energy (CVE)] if any state of the system can be steered to the origin (respectively, any state) with an arbitrarily small amount of energy in the L_2 (square integral) sense as the time duration becomes large. A time-invariant system (A,B) is NCVE (CVE) if and only if (A,B) is controllable and all eigenvalues of A have nonnegative (respectively, zero) real parts [15,16]. A periodic system $[A(t),B(t)]$ is NCVE (CVE) if and only if it is controllable on some interval and all eigenvalues of the monodromy matrix lie inside (on) the unit disk [17]. The latter can be replaced by the condition that all eigenvalues of $\bar{A}$ have nonnegative (zero) real parts. Here, $\bar{A}$ is the constant matrix in the Floquet representation. Thus, the following theorem holds.

Theorem 2. System (10) is NCVE (CVE).

Proof. In view of Theorem 1, all eigenvalues of the matrix $\bar{A}$ have zero real parts. Hence, $[A(t),B]$ is NCVE (CVE). This is also a direct consequence of Eq. (18).

Now special cases of transverse spin and pure spin are considered. First, assume $\bar{\boldsymbol{\omega}}=[\Omega_1,0,0]'$. Then the reference motion is a transverse (or flat) spin, and the set of eigenvalues of the system matrix A_{tra} is

$$\sigma(A_{\text{tra}})=\{0(\text{quadruple}),\pm i\Omega_1\}$$

Thus, NCVE (CVE) of (A_{tra},B) is confirmed. There is a single angular frequency Ω_1. The case $\bar{\boldsymbol{\omega}}=[0,\Omega_2,0]'$ is physically the same because the satellite is symmetric with respect to the $\boldsymbol{b}_3$ axis. The set of eigenvalues of the system matrix $\tilde{A}_{\text{tra}}$ in the general case $\bar{\boldsymbol{\omega}}=[\Omega_1,\Omega_2,0]'$ is

$$\sigma(\tilde{A}_{\text{tra}})=\{0(\text{quadruple}),\pm i\Omega_{12}\}$$

By replacing the $\boldsymbol{b}_1$ axis by $\tilde{\boldsymbol{b}}_1$ along $\bar{\boldsymbol{\omega}}$, this case is reduced to the transverse spin with spin rate Ω_{12} about the $\tilde{\boldsymbol{b}}_1$ axis.

Consider the case $\bar{\boldsymbol{\omega}}=[0,0,\Omega_3]'$. This is a pure spin about the axis of symmetry $\boldsymbol{b}_3$, and the set of eigenvalues of the system matrix A_{pur} is

$$\sigma(A_{\text{pur}})=\{0(\text{double}),\pm i\Omega_3,\pm ik\Omega_3\}$$

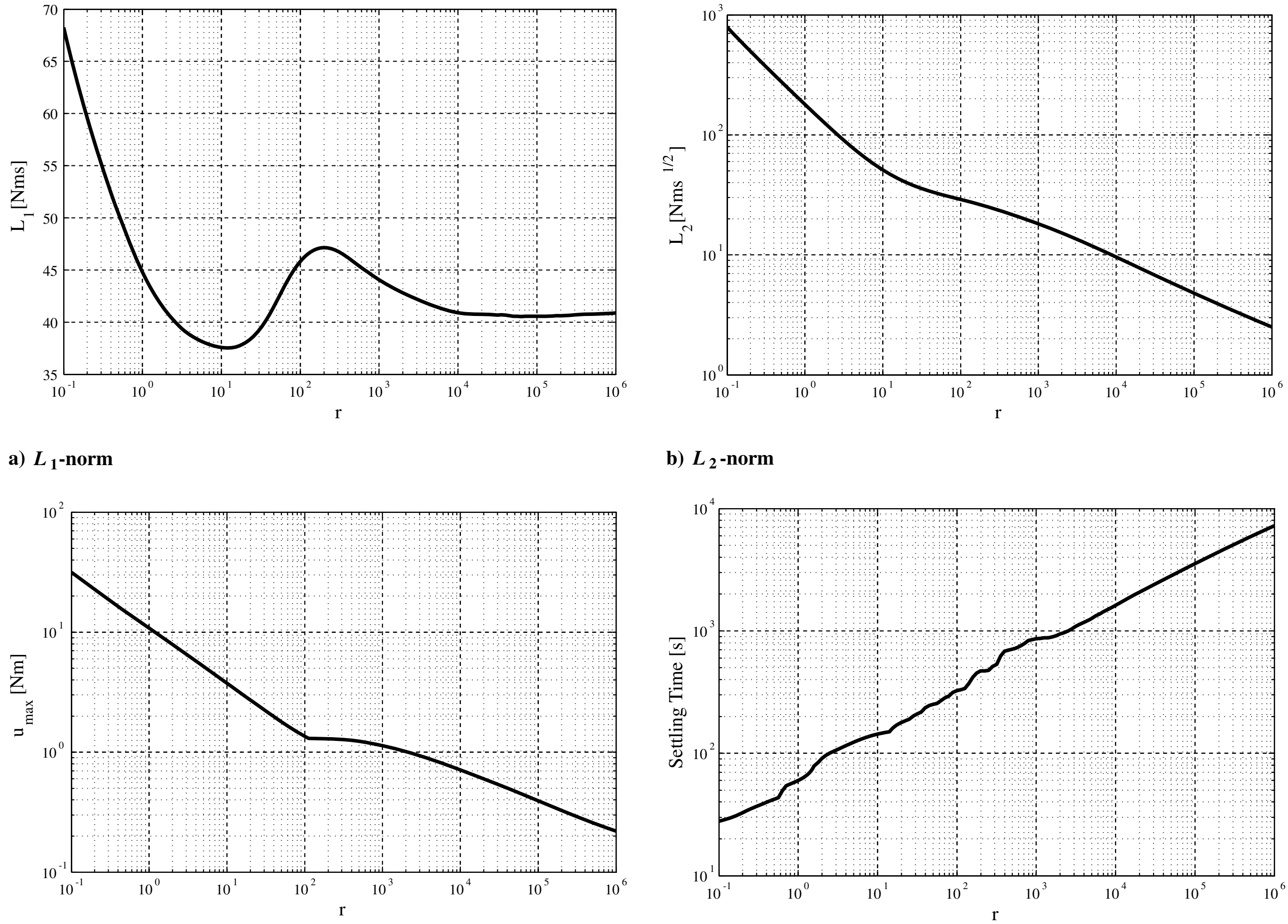
In this case, there are two angular frequencies: the spin rate Ω_3 and the transverse frequency $\mu=k\Omega_3$.

If $J_1=J_2=J_3$, it is a special case of a symmetric satellite, and then $k_i=0$ and $\dot{\bar{\boldsymbol{\omega}}}=0$. Thus, $\bar{\boldsymbol{\omega}}=[\Omega_1,\Omega_2,\Omega_2]'$. In this case,

$$\sigma(A_{\text{sym}})=\{0(\text{quadruple}),\pm i\Omega\}$$

where A_{sym} is the system matrix, and $\Omega=(\Omega_1^2+\Omega_2^2+\Omega_3^2)^{1/2}$.

Some remarks on asymmetric satellites are in order. If the reference motion is a spin about a principal axis, A is constant. If it is about a major or minor axis, (A,B) is NCVE (CVE). However, if the



a) L_1-norm

b) L_2-norm

c) Maximum value of the input

d) Settling time

Fig. 1 Performance indices vs *r*: general spin.

spin is about an intermediate axis, A is unstable, and (A, B) fails to have NCVE. These results are easily derived from Eq. (6).

If $\bar{\omega} = [0, 0, 0]'$, the reference motion is stationary, and it covers the reorientation problem of a satellite. In this case, $\sigma(A_{\rm sta}) = \{0(\text{sextuple})\}$.

Feedback Control with Minimum L_1-Norm

The NCVE property of system (10) indicates that the L_2-norm of the input torque is not an appropriate performance index because it does not represent fuel consumption. The right performance index is the L_1-norm as explained in the Inroduction, but there are no practical theories that are directly related to L_1-norm. Fortunately, the LQR theory still provides us feedback controls with small L_1-norm. To see this, the optimal regulator problem is briefly reviewed.

Consider system (10),

$$\dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + B\boldsymbol{u}(t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0$$

and the optimal regulator problem defined by the quadratic cost function:

$$J(u; x_0) = \int_{t_0}^{\infty} [x'(t)Qx(t) + u'(t)Ru(t)]\mathrm{d}t \tag{19}$$

where $Q \geq 0$, and $R > 0$. Because (A, B) is controllable, for each Q with $(Q^{1/2}, A)$ detectable, there exists a unique nonnegative p-periodic solution to the differential Riccati equation (DRE):

$$-\dot{X} = A'X + XA - XBR^{-1}B'X + Q \tag{20}$$

such that $A - BR^{-1}B'X$ is exponentially stable [19]. This solution is referred to as the stabilizing solution. The optimal control is given by the feedback control:

$$\boldsymbol{u}^*(t) = -R^{-1}B'X(t)\boldsymbol{x}(t) \tag{21}$$

and the minimum cost is given by $J(u^*, x_0) = x_0'Xx_0$. Recall that a periodic system $[A(t), B(t)]$ is NCVE if and only if it is controllable on some interval and $X = 0$ is the only nonnegative p-periodic

Table 1 Performance indices: general spin

Parameters	Values
r	12.46
L_1-norm	37.51, Nms
L_2-norm	46.91, Nms$^{1/2}$
$u_{\max}$	3.388, Nm
Settling time T	147.3 s

Table 2 Performance indices: pure spin

Parameters	Values
r	29.73
L_1-norm	49.35, Nms
L_2-norm	95.35, Nms$^{1/2}$
$u_{\max}$	4.563, Nm
Settling time T	196.9 s

solution of the DRE with $Q = 0$. This is a second set of necessary and sufficient conditions in [17]. Thus, the solution $X \to 0$ as $Q \to 0$, and feedback controls with arbitrarily small L_2-norm can be designed by choosing small Q or, alternatively, by making R relatively larger than Q. For relative orbit transfer problems in [16], the L_1-norm of the feedback control decreases monotonically as R increases. Hence, it is used as a design parameter of feedback controls with small L_1-norm. For rotational motion control, the L_1-norm is not monotonically decreasing, but R is still an effective design parameter, as shown by numerical simulations in the next section. In the special case of a transverse or a pure spin, the matrix A is constant. The DRE is then replaced by the algebraic Riccati equation (ARE),

$$A'X + XA - XBR^{-1}B'X + Q = 0$$

and the feedback control is given by

$$\boldsymbol{u}^*(t) = -R^{-1}B'X\boldsymbol{x}(t) \tag{22}$$

In view of Eq. (16), the feedback control (21) can be replaced by

$$\bar{\boldsymbol{u}}(t) = -P_0(\mu t)R^{-1}B'\bar{X}P(\mu t)\boldsymbol{x}(t) \tag{23}$$

where $\bar{X}$ is the stabilizing solution of the ARE:

$$\bar{A}'\bar{X} + X\bar{A} - \bar{X}BR^{-1}B'\bar{X} + Q = 0 \tag{24}$$

Assume that weight matrices Q and R are diagonal and given by

$$Q = \mathrm{diag}\{r_1 I_{3\times3}, \quad r_2 I_{3\times3}\}$$

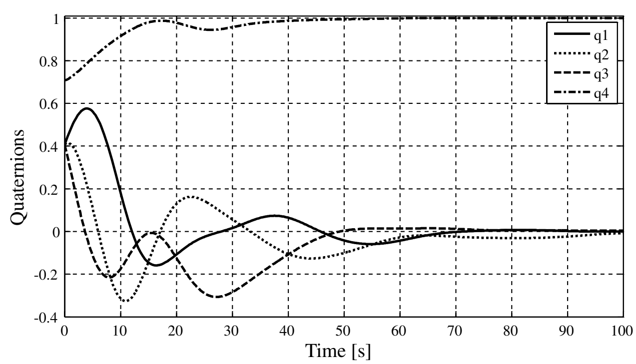
and $R = rI_{3\times3}$. In this case, the following is true, and $\boldsymbol{u}^*$ and $\bar{\boldsymbol{u}}$ coincide.

Theorem 3. The p-periodic stabilizing solution of the DRE is given by $X(t) = P(t)'\bar{X}P(t)$, where $\bar{X}$ is the stabilizing solution of the ARE [Eq. (24)].

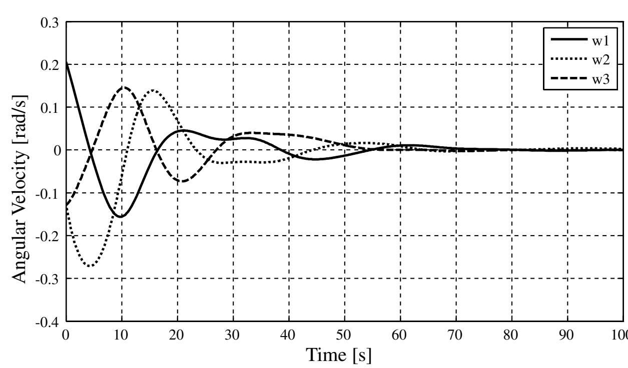
Proof. Note that $P(t)$ commutes with Q and $BR^{-1}B'$. Then direct differentiation of $X(t) = P(t)\bar{X}P(t)'$, together with the ARE and the equality

$$\bar{A} = P'(t)[A(t)P(t) - \dot{P}(t)]$$

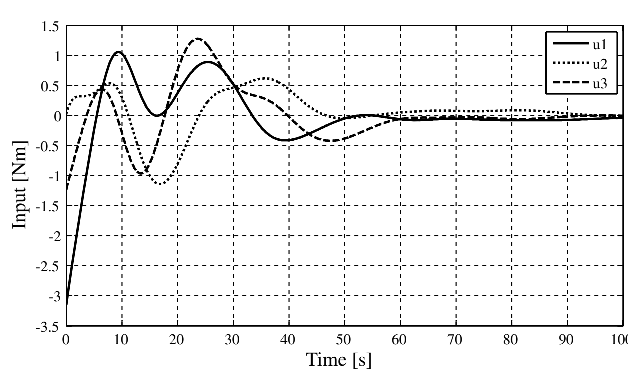
show that it satisfies the DRE.



a) Quaternions

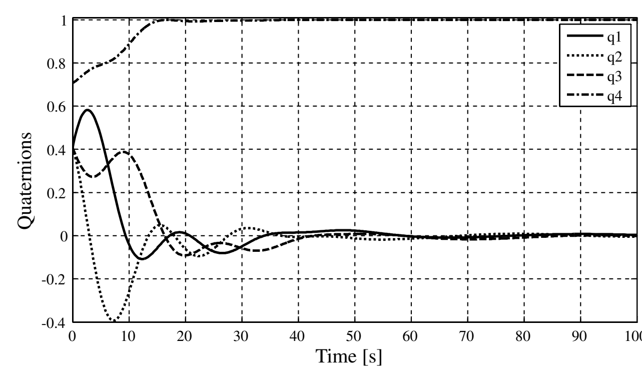


b) Angular rates

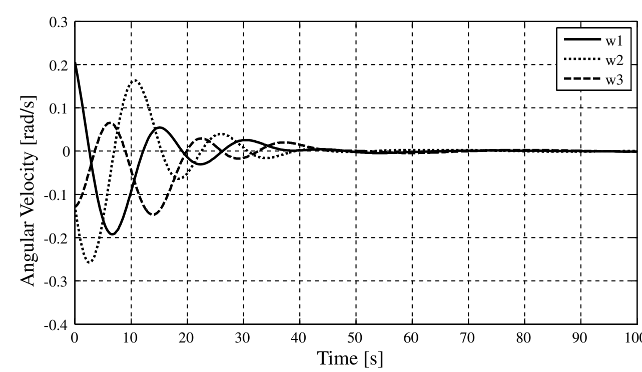


c) Input torque

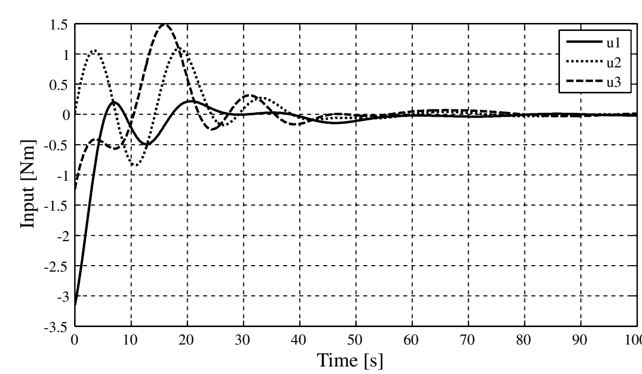
Fig. 2 Nonlinear system: general spin.



a) Quaternions

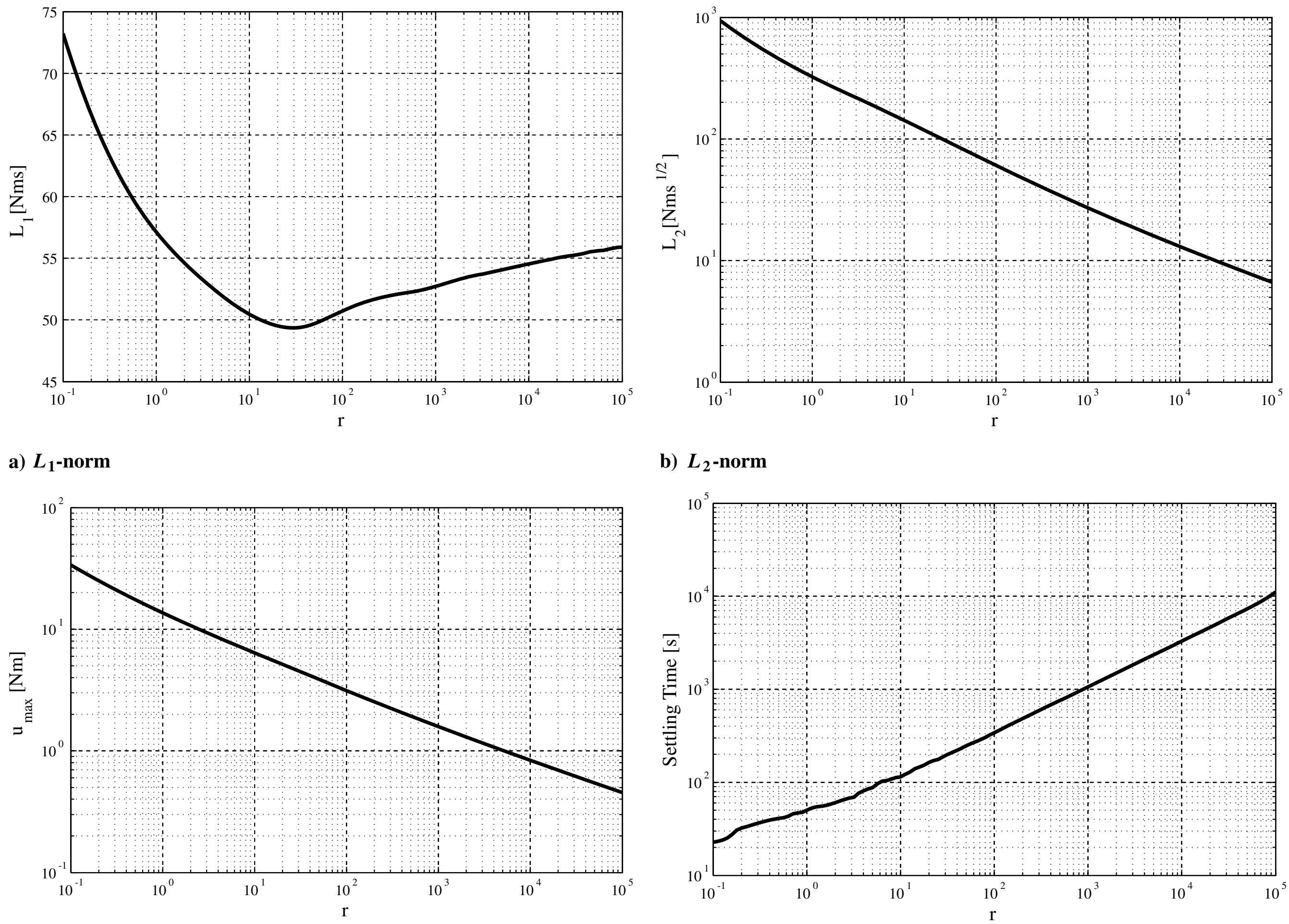


b) Angular rates



c) Input torque

Fig. 3 Linear system: general spin.



c) Maximum value of the input

d) Settling time

Fig. 4 Performance indices vs r: pure spin.

Thus, in all cases, feedback controls can be constructed from AREs. As shown in the next section, the L_1-norm of the feedback control decreases as r increases up to a certain value. It is not monotonically decreasing, but it has a global minimum. Hence, feedback controls with minimum L_1-norm can be easily designed.

Simulation Results

For simulations, the inertia matrix is set:

$$J = \text{diag}\{100, \quad 100, \quad 50\}\ \text{kgm}^2$$

This yields $k_1 = -k_2 = k = 0.25$ and $k_3 = 0$. The initial quaternions of the reference motion are assumed to be $\bar{\boldsymbol{q}}(0) = [0, 0, 0, 1]'$ so that $\bar{R}_B$ and the reference frame R_I initially coincide. The initial condition $\boldsymbol{q}(0)$ is determined by the $\pi/2$ (rad) rotation of R_I about the Euler axis $(1/3^{1/2})[1, 1, 1]'$. The initial angular velocity is set to $\boldsymbol{\omega}(0) = 0.9\bar{\boldsymbol{\omega}}(0)$, and $\bar{\boldsymbol{\omega}}(0)$ is chosen from the angular momentum ellipsoid

$$\sum_i J_i^2 \omega_i^2 = H^2$$

with $H = 10\pi$ Nms.

Parameters of Q are set to $r_1 = 200$ and $r_2 = 1$, because Eq. (3) for $\boldsymbol{q}_r$ is driven by $\boldsymbol{\omega}_r$, and it is desirable for $\boldsymbol{q}_r$ to converge to zero faster than $\boldsymbol{\omega}_r$. To design feedback controls with a small L_1-norm, the parameter r of the matrix R is varied. Feedback controls given by Eq. (21) are then applied to the original nonlinear system (3) and (4), and the L_1-norm of the controller and the settling time are calculated. The settling time is defined as the first time after which all the elements of the state x have a modulus less than 10^{-4}. To see NCVE, the L_2-norm of the input torque is also given.

In the relative orbit control problem associated with HCW or TH, Eq. (16), L_1- and L_2-norms decrease monotonically as r increases. However, for rotational motion control, the L_1-norm is not monotone in r. Instead, it has a global minimum. This is a new feature of the rotational motion. The weight r, which gives the minimum, is regarded as optimal. Then feedback controls are designed and applied to the nonlinear system (3) and (4), and simulation results are given. For comparison, they are also applied to the linear system. In this case, q_{r4} is calculated by

$$q_{r4} = (1 - x_1^2 - x_2^2 - x_3^2)^{1/2}$$

Note that various suboptimal controls with shorter settling time can be designed for smaller r.

As a first example, a general spin with

$$\boldsymbol{\omega}(0) = \pi[\sqrt{3}/20, \quad 0, \quad 1/15]$$

is considered. Then the reference motion is

$$\bar{\boldsymbol{\omega}}(t) = (\pi/120)[6\sqrt{3}\cos(\pi/60)t, \quad 6\sqrt{3}\sin(\pi/60)t, \quad 8]'$$

In this case, $\Omega_3 = \pi/15$, $\mu = k\Omega_3 = \pi/60$, and the period is 120 s. The initial conditions are given by

$$\boldsymbol{q}_r(0) = [0.4082, \quad 0.4082, \quad 0.4082, \quad 0.7071]'$$

and

$$\boldsymbol{\omega}_r(0) = [0.2053, \quad -0.1243, \quad -0.1291]'$$

The performance indices of the nonlinear system against the parameter r are given in Figs. 1a–1d, and those for the optimal r are shown in Table 1. The L_1-norm has a clear global minimum 37.51 at $r = 12.46$. The responses of the nonlinear and linear systems to the optimal feedback and control inputs are depicted in Figs. 2a–2c and 3a–3c, respectively. The settling time is 147 s. Any $3 \leq r \leq 30$ is suboptimal, and the settling time can be reduced by choosing a smaller r. If an upper bound of the admissible L_1-norm and the settling time are specified, they determine a range of the values of the parameter r, and a set of feedback controllers can be designed.

As a second example, consider $\bar{\boldsymbol{\omega}}(0) = [0, 0, 2\pi/15]'$, which is a pure spin about the axis of symmetry $\bar{\boldsymbol{b}}_3$ with a period of 15 s. This is the reorientation problem of a spinning spacecraft. In view of Eq. (1),

$$\bar{q}_1(t) = \bar{q}_2(t) = 0, \quad \bar{q}_3(t) = \sin(2\pi/15)t, \quad \bar{q}_4(t) = \cos(2\pi/15)t$$

Initial conditions are then given by

$$\boldsymbol{q}_r(0) = [0.4082, \quad 0.4082, \quad 0.4082, \quad 0.7071]'$$

and

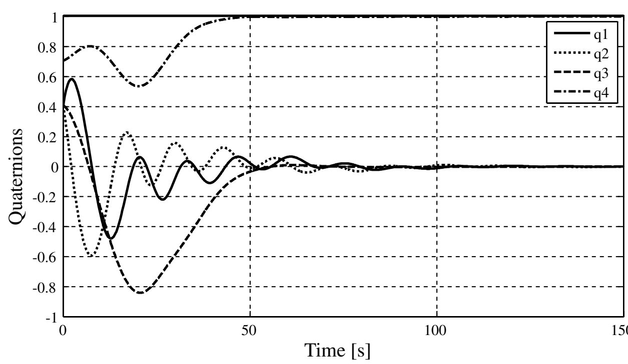
$$\boldsymbol{\omega}_r(0) = [0.1022, \quad -0.3815, \quad 0.2374]'$$

The performance indices against r are depicted in Figs. 4a–4d, and the minimum L_1-norm is $L_1 = 49.35$ at $r = 29.73$. The performance indices for the optimal r are shown in Table 2, and the responses of the nonlinear and linear systems to the optimal feedback and control inputs are depicted in Figs. 5a–5c and 6a–6c, respectively. The settling time is 197 s. Any $20 \leq r \leq 90$ can be regarded as suboptimal.

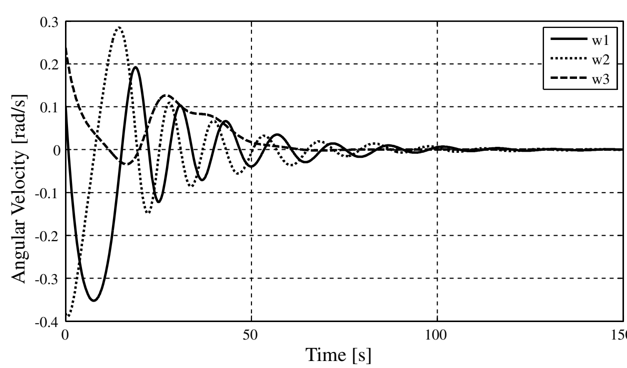
Simulations are performed for various other initial conditions, but the results are similar. Simulations in the case of a transverse spin are also done, and results similar to those of the pure spin are obtained. For rest-to-rest attitude control problems, the proposed controller also gives good performance.

Conclusions

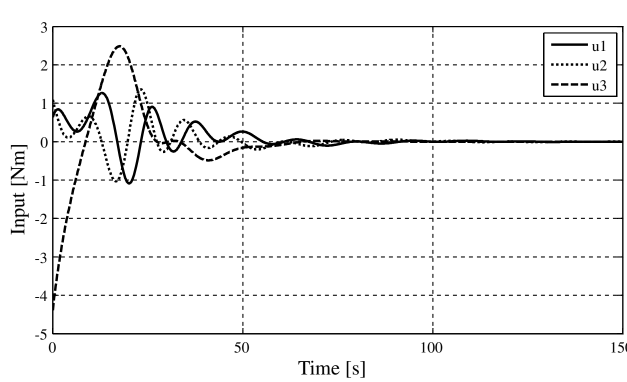
Rotational motion control of a spacecraft to a given reference motion has been considered. For the linearized system of the relative rotational motion, the Floquet representation of its state transition



a) Quaternions

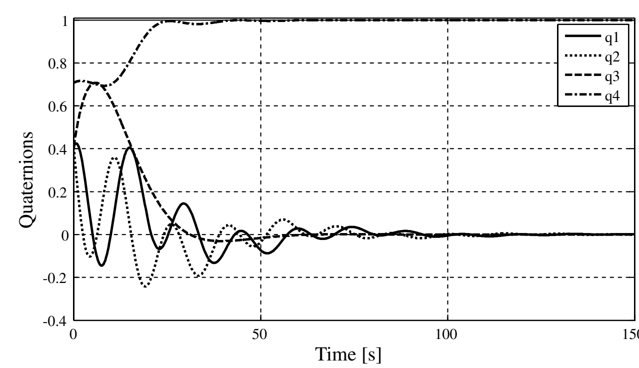


b) Angular rates

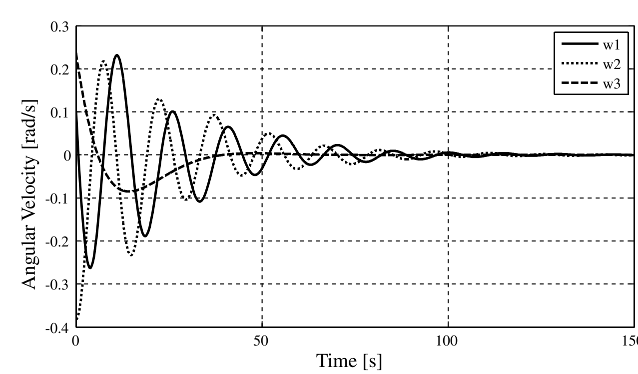


c) Input torque

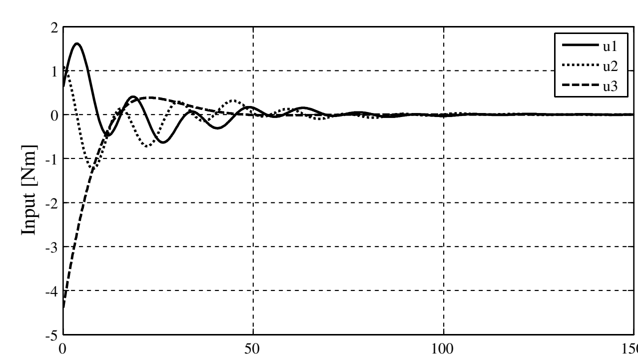
Fig. 5 Nonlinear system: pure spin.



a) Quaternions



b) Angular rates



c) Input torque

Fig. 6 Linear system: pure spin.

matrix is explicitly given, and the property of null controllability with vanishing energy is proved. Based on these results, L_1 optimal and suboptimal feedback controllers are designed by the LQR theory. Simulation results show that the proposed controllers give good performances for various types of rotational motion control problems.

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